particle. We then replace $F_{N,N+1}$ by $p(x_0+\Delta x_0/2)$, $F_{N-1,N}$ by $p(x_0-\Delta x_0/2)$, S_N by $S(x_0,T)$, and unit mass by $\rho_0\Delta x_0$. Then Eq. (3) becomes

$$\rho_0\Delta x_0 d^2S/dT^2=-p[(x_0+\Delta x_0)/2]+p[(x_0-\Delta x_0)/2]+\eta[u(x_0+\Delta x_0)-2u(x_0)+u(x_0-\Delta x_0)],$$

where $u=dS/dT$. In the limit as $\Delta x_0\to 0$,

$$\frac{d^2S}{dT^2}=-\frac{1}{\rho_0}\frac{\partial p}{\partial x_0}+\frac{\mu}{\rho_0}\frac{\partial^2 u}{\partial x_0^2}, \quad (4)$$

where $\mu=\eta\Delta x_0$ is the ordinary viscosity and p is compressive stress in the x_0 direction. Equation (4) is the equation of motion of a continuum in uniaxial strain. In the dimensionless Eq. (3) the mass of each particle is unity and $\Delta x_0=1$, therefore, $\rho_0=1$, $\mu=\eta$, and x_0 is dimensionless.

To further relate the lattice under study to the continuum, we define as strain between the N and $N+1$ mass points $\epsilon_N=S_{N+1}-S_N$, since the undisturbed separation is taken as unity. The continuum analogue is $\epsilon=(V-V_0)/V_0$, where V_0 is undisturbed specific volume. Then Eqs. (1) and (2) become

$$-F_{N,N+1}=\epsilon_N-\alpha\epsilon_N^2$$
$$-G_{N,N+1}=\eta\epsilon_N'. \quad (5)$$

Equations (5) go directly into the form assumed by Bland in calculating steady profiles[3]

$$-p-q=\sigma_x=\epsilon-\alpha\epsilon^2+\eta\epsilon'$$
$$-q=\eta\partial u/\partial x_0=\eta\epsilon'. \quad (6)$$

In the continuum case Eq. (4) is supplemented by an equation of conservation of mass:

$$\partial x/\partial x_0=\rho_0/\rho=V, \quad (7)$$

where x is the Euler coordinate. In the lattice problem mass conservation is assured by the assumption that each lattice point is occupied by a constant mass.

III. SHOCK PROFILE WITH DISSIPATION

The permanent regime or steady profile is obtained in the continuum case by solving Eqs. (4) and (7) subject to the conditions that $(\partial/\partial T)_x\equiv 0$, that a mass element approaches a uniform undisturbed state as $t\to-\infty$, and a uniform compressed state as $t\to+\infty$.[2] A condition equivalent to the first of these is that p, u, ρ be functions only of the variable $\xi=t-\theta x$, where θ is to be determined by the boundary conditions

$$du/d\xi\to 0,\ p\to p_0,\ u\to 0,\ \rho\to\rho_0 \quad \text{as} \quad \xi\to-\infty$$
$$du/d\xi\to 0,\ p\to p_1,\ u\to u_1,\ \rho\to\rho_1 \quad \text{as} \quad \xi\to+\infty. \quad (8)$$

In either case we arrive at a relation

$$\theta^2\eta du/d\xi=(1-\theta^2)u-\alpha\theta^3u^2, \quad (9)$$

where the term on the left represents the viscous force. The boundary conditions of Eq. (8) imply that

$$u_1=(1-\theta^2)/\alpha\theta^3. \quad (10)$$

Equation (9) can be integrated directly, and if the origin, $\xi=0$, is chosen where $u=u_1/2$, we obtain

$$u=(u_1/2)[1+\tanh(u_1\alpha\theta\xi/2\eta)], \quad (11)$$

where $U_s=1/\theta$ is shock velocity. Note particularly that when viscosity vanishes, the shock profile becomes a discontinuity in u, p, ρ, etc.

In the lattice problem we proceed in a similar manner. Combining Eqs. (1)–(3) produces the equation

$$S_N''(T)=(S_{N+1}-2S_N+S_{N-1})\{1-\alpha(S_{N+1}-S_{N-1})\}+\eta(S_{N+1}'-2S_N'+S_{N-1}'), \quad (12)$$

where each displacement is evaluated at time T. We again seek progressing wave solutions in the form

$$S_N(T)=S(T-N\theta)\equiv S(\xi). \quad (13)$$

If we define $D\equiv d/d\xi$, then

$$S_N'(T)=DS,$$
$$S_{N+1}(T)=S(\xi-\theta)=e^{-\theta D}S(\xi),$$
$$S_{N-1}(T)=S(\xi+\theta)=e^{\theta D}S(\xi). \quad (14)$$

Equation (12) then becomes the ordinary differential equation,

$$D^2S=[2(\cosh\theta D-1)S][1+2\alpha(\sinh\theta D)S]+\eta D[2(\cosh\theta D-1)S]. \quad (15)$$

By expanding the operators in series and keeping only the lowest order terms, we obtain the equation

$$D^2S=\theta^2D^2S+\theta^4D^4S/12+\cdots+2\alpha\theta^3DS\cdot D^2S+\cdots+\eta\theta^2D^3S+\cdots. \quad (16)$$

If we discard terms of fourth order or higher and substitute $u\equiv DS$, $u'=Du$, etc, Eq. (16) becomes

$$(1-\theta^2)u'=2\alpha\theta^3uu'+\eta\theta^2u''. \quad (17)$$

Integrating once

$$(1-\theta^2)u=\alpha\theta^3u^2+\eta\theta^2u'+A. \quad (18)$$

Boundary conditions are the same as those in Eq. (8):

$$\xi\to-\infty,\ u'\to 0,\ u\to 0$$
$$\xi\to+\infty,\ u'\to 0,\ u\to u_1. \quad (19)$$

Applied to Eq. (18), these yield the results $A=0$, $u_1=(1-\theta^2)/\alpha\theta^3$, as for the continuum case. Since Eqs. (9) and (18), with $A=0$, are identical with identical boundary conditions, we conclude that they lead to the same shock profile, Eq. (11), and to the same relation between shock and particle velocity, Eq. (10).

The present calculation has been carried out with a particular force law, Eq. (5), but we infer the following: In the presence of dissipation there is no distinction

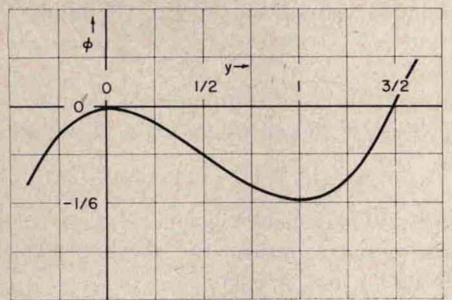

FIG. 4. Potential of Eq. (24).

TABLE II. Results of numerical integration of equation (12); $\eta=0$.

Computation number	α	u_1	$u_1\alpha$	θ	$(\theta/12\, u_1\alpha)^{1/2}$
P-26	0.1	0.1	0.01	0.99506	2.88
P-27	0.3	0.1	0.03	0.98554	1.66
P-28	1.0	0.1	0.1	0.95540	0.90
P-25	3.0	0.1	0.3	0.88856	0.50
P-32	10.0	0.1	1.0	0.75488	0.25

between the one-dimensional lattice and the continuum in uniaxial strain, except that the Lagrangian space variable x_0, is replaced by N, provided that higher-order derivatives in Eq. (16) are neglected.

IV. SHOCK PROFILE WITHOUT DISSIPATION

In the continuum, we noted that the shock transition becomes a discontinuity in the absence of rate-dependent forces. The same was found to be true of a damped lattice if the series of Eq. (16) was truncated at the third-order term. In both these cases, the shock transition is a smooth region of monotonic transition from the uncompressed initial state to the compressed final state. If, however, we include higher-order terms of Eq. (16), allowing η to become very small or vanish, a finite transition region remains. This can be illustrated by retaining the fourth-order term in Eq. (16). Then in place of Eq. (18) we have,

$$(1-\theta^2)u=\theta^4u''/12+\alpha\theta^3u^2+\eta\theta^2u'+A. \qquad (20)$$

The boundary conditions of Eq. (19) are now inadequate to evaluate A. We can, however, extend these boundary conditions on physical grounds. If we concede that any disturbance in a real lattice propagates at finite velocity, then a mass point ahead of the disturbance is not only at rest. It undergoes no acceleration until the disturbance reaches it; and in fact it is absolutely quiescent in the model we assume here, so that all derivatives vanish. This means that the series of Eq. (16) can be extended to arbitrarily high derivatives, and A will always vanish.

TABLE I. Amplitude dispersion of Eq. (25).

Amplitude a	ω: Numerical integration	ω: Eq. (28)	Period, $\Delta\tau=2\pi/\omega$
0	···	1.000	6.28
0.25	0.982	0.974	6.40
0.5	0.911	0.896	6.90
0.75	0.785	0.766	8.00

We consider, then, Eq. (20) with $A=0$:

$$(\theta^4/12)u''=(1-\theta^2)u-\alpha\theta^3u^2-\eta\theta^2u'. \qquad (21)$$

This is the equation of a damped oscillator. As $\xi\rightarrow\infty$ it comes to equilibrium and the derivatives of u become arbitrarily small. We can then evaluate u_1 by the condition that, as $\xi\rightarrow\infty$,

$$u\rightarrow u_1,\ u'\rightarrow 0,\ u''\rightarrow 0. \qquad (22a)$$

Then, as before,

$$u_1=(1-\theta^2)/\alpha\theta^3. \qquad (22b)$$

In discussing Eq. (21), it is useful to make the transformation $u=u_1y$, $\xi=\theta^2\tau/[12(1-\theta^2)]^{1/2}$. Then Eq. (21) becomes

$$d^2y/d\tau^2=y-y^2-\eta[12/(1-\theta^2)]^{1/2}(dy/d\tau). \qquad (23)$$

This is the equation of a particle moving, with damping, in a potential,

$$\phi(y)=y^3/3-y^2/2, \qquad (24)$$

shown in Fig. 4. It has a maximum at $y=0$, a minimum at $y=1$, and it goes monotonically to infinity outside these limits. By virtue of the boundary conditions at $\xi\rightarrow-\infty$, viz. $u=u'=0$, the initial position of this "pseudoparticle" is at the maximum $\phi=0$, $y=0$. When the shock arrives, the pseudoparticle is moved to the right by the first infinitesimal disturbance and it

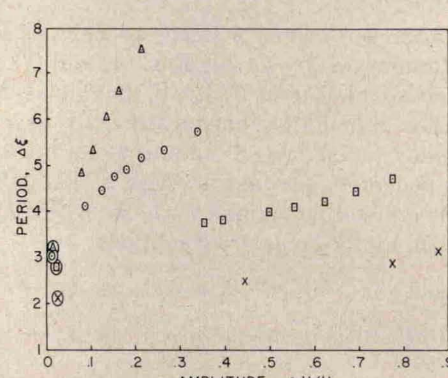

FIG. 5. Effects of $u_1\alpha$ and amplitude on period, Eq. (12): △—$\alpha=0.1$, $u_1=0.1$, $N=90$; ⊙—$\alpha=1.0$, $u_1=0.1$, $N=90$; ⊡—$\alpha=3.0$, $u_1=0.1$, $N=90$; ×—$\alpha=10.0$, $u_1=0.1$, $N=90$; for circled data points, $N=30$